Design of Quadratic Optimal Regulator for DC Motor

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Abstract

This paper deals with the mathematical modeling of a dc motor and quadratic optimal regulator system. MATLAB based program is used to design the quadratic optimal regulator system for the dc motor.

Key words: dc motor, quadratic optimal regulator, mathematical modeling, MATLAB.

1. Introduction

We know that it is possible to design state feedback controllers using using eigenvalue (pole) placement algorithms. For single input systems, given a set of desired eigenvalues, the feedback gain to achieve this is unique (as long as the system is controllable). How does one utilize this freedom? A more fundamental issue is that the choice of eigenvalues is not obvious. For example, there are tradeoffs between robustness, performance, and control effort.

Linear quadratic (LQ) optimal control can be used to resolve some of these issues, by not specifying exactly where the closed loop eigenvalues should be directly, but instead by specifying some kind of performance objective function to be optimized.

The system that minimizes the selected performance index is, by definition optimal. Although the controller may have nothing to do with "optimality" in many practical applications, the important point is that the design based on the quadratic performance index yields a stable control system.[5],[6],[7].

This paper is mainly discussing about the modeling of dc motor [1], [2], and Quadratic Optimal Regulator. MATLAB based analysis is also shown at the end.

2. Mathematical Modeling of DC Motor

The different equations related to DC Motor are given below:

$$\frac{dx_1(t)}{dt} = x_2(t) \quad (1)$$

$$\frac{dx_2(t)}{dt} = -\frac{B}{J}x_2(t) + \frac{K_t}{J}x_3(t) \quad (2)$$

$$\frac{dx_3(t)}{dt} = -\frac{K_m}{L_m}\, x_2(t) - \frac{R_m}{L_m}\, x_3(t) + \frac{1}{L_m}e_a \quad (3)$$

$$y(t) = \theta(t) = x_1(t) \quad . \quad (4)$$

We chose the different state variable of the DC Motor as $x_1(t) = \theta(t)$, $x_2(t) = \omega(t) = \frac{d\theta(t)}{dt}$ and $x_3(t) = i_a(t)$.

Where $e_a(t)$ = armature voltage, $i_a(t)$ = armature current, $\theta(t)$ = motor shaft angle, $\frac{d\theta(t)}{dt} = \omega(t)$= shaft speed, J = moment of inertia of the rotor, B = viscous frictional constant, L_m= inductance of armature windings, R_m= armature winding resistance, K_t= motor torque constant, K_m= motor constant.
Here the motor speed ω (*t*) is controlled by varying the armature voltage e_a(t). Hence $e_a(t)$ is the input variable and ω (*t*) is the output variable.

Hence the state model of DC Motor is derived from equations (1), (2), (3) and (4) as follows

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dx_3(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{B}{J} & \frac{K_t}{J} \\ 0 & -\frac{K_m}{L_m} & -\frac{R_m}{L_m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L_m} \end{bmatrix} u(t) \quad (5)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad (6)$$

Let, the motor parameters (coefficient of differential equations) are assigned to be $L_m = 0.5$ H, K_t = 0.01 N-m/A, K_m= 0.01 V-sec/rad, J = 0.01 kg-m2, B = 0.1 N-m-sec/rad, R_m= 1 Ω.

Thus the state model of dc motor is derived using motor parameters and equation (5) and (6) as follows:

$$\begin{bmatrix} \frac{dx_1(t)}{dt} \\ \frac{dx_2(t)}{dt} \\ \frac{dx_3(t)}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -10 & 1 \\ 0 & -0.02 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u(t) \quad (7)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad (8)$$

3. Mathematical modeling of Quadratic Optimal Regulator

The given system equation is

$\dot{x} = Ax + Bu$ (9)

This determines the matrix K of the optimal control vector

$u = -Kx$ (10)

So as to minimize the performance index

J = $\int_0^{\infty}(X^T\mathrm{Q}X + u^T\mathrm{R}u)\,dt$ (11)

Where Q is a positive definite (positive semi definite) Hermitian or real symmetric matrix and R is a positive-definite Hermitian or real symmetric matrix.

It can be proved that if A-BK is a stable matrix, there exists a posite definite matrix P that satisfies the following equation:

$(A - BK)^T$P+P(A–BK)=-(Q+K^TRK) (12)

Since all eigen values of A-BK are assumed to have negative real parts, we have x(∞) → 0, therefore we can write

J = X^T(0)PX(0) (13)

Thus performance index J can be obtained in terms of the initial condition X (0) and P.

Thus the optimal matrix K is given by

K = $R^{-1}B^T$P (14)

The matrix in equation (14) must satisfy equation (12) or the following equation:

A^TP+PA-PB$R^{-1}B^T$P+Q=0 (15)

The above equation is called the reduced –matrix Riccati equation.

The design steps may be stated as follows:

a) Solve equation (15), the reduced Riccati equation for matrix P.
b) Substititute this matrix P into equation (14). The resulting matrix K is the optimal matrix.

4. MATLAB Based Quadratic Optimal Regulator for dc motor

The given MATLAB command is used to solve the continuous time, linear, quadratic regulator problem and the associated Riccati equation.

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>> A = [0 1 0;0 -10 1;0 -0.02 -2];

B = [0;0;2];

Q = [1 0 0;0 1 0;0 0 1];

R = [ 1];

[K,P,E] = lqr(A,B,Q,R)

K =

    1.0000    0.1069    0.4515

P =

   14.6323    1.4515    0.5000

    1.4515    0.1945    0.0535

    0.5000    0.0535    0.2258

E =

   -0.0705

   -2.8373

   -9.9952
```

5. Conclusion

The result has shown the gain matrix K, eigenvalue vector E, and matrix P of the quadratic optimal regulator system for dc motor because matrix A-BK is a stable matrix. It is important to note that for certain systems matrix A-BK cannot be made a stable matrix, whatever K is chosen. In such a case, there does not exist a positive-definite matrix P for the matrix Riccati equation.

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